Characterization of Poisson integrals on non-tube bounded symmetric domains

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CHARACTERIZATION OF POISSON INTEGRALS FOR NON-TUBE BOUNDED SYMMETRIC DOMAINS

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ABSTRACT. We characterize the L^p-range, $1 < p < +\infty$, of the Poisson transform on the Shilov boundary for non-tube bounded symmetric domains. We prove that this range is a Hua-Hardy type space for harmonic functions satisfying a Hua system.

1. INTRODUCTION

Let $\Omega = G/K$ be a Riemannian symmetric space of non-compact type. To each boundary G/P one can define a Poisson transform, which is an integral operator from hyperfunctions on G/P into the space of eigenfunctions on Ω of the algebra $\mathcal{D}(\Omega)^G$ of invariant differential operators. For the maximal boundary, $G/P_{\min}$, the most important result is the Helgason conjecture, proved by Kashiwara *et al.* [8] which states that a function is eigenfunction of all invariant differential operators on Ω if and only if it is Poisson integral

$$\mathcal{P}_\lambda f(gK) = \int_K f(k)e^{-\langle\lambda+\rho,H(g^{-1}k)\rangle}dk.$$

of a hyperfunction on the maximal boundary, for a generic $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. For other function spaces such as $L^p(G/P_{\min})$ the characterization is in connection with Fatou's theorems. We mention here the work of Helgason [5] and Michelson [14] for $p = \infty$, and Sjörgen [17] for $1 \leq p < \infty$ using weak L^p-spaces. Another characterization for $1 \leq p \leq \infty$, using Hardy-type spaces, was done by Stoll [18] in the harmonic case and by Ben Saïd *et al.* [1] in the general case.

If Ω is a bounded symmetric domain, one is interested in functions whose boundary values are supported on the Shilov boundary (minimal boundary) $S := G/P_{\max}$ rather that the maximal boundary $G/P_{\min}$.

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For the Shilov boundary the Poisson transform is defined by

$$\mathcal{P}_s f(gK) = \int_K f(k) e^{-\langle s\rho_0 + \rho_1, H_1(g^{-1}k)\rangle} dk, \quad s \in \mathbb{C}.$$

In this case, Hua [6] had proved that the algebra of invariant differential operators is not necessarily the most appropriate for characterizing harmonic functions (i.e., annihilated by the algebra $\mathcal{D}(\Omega)^G$). Johnson and Koràny i [7], generalizing the earlier work of Hua, Korányi and Stein [10], and Korányi and Malliavin [11], introduced an invariant second order ($\mathfrak{k}_{\mathbb{C}}-$valued) operator $\mathcal{H}$, called since, second-order Hua operator (or Hua system). They showed, in the tube case, that a function is annihilated by the Hua operator if and only if it is the Poisson integral $\mathcal{P}_{s_0} f$ $(s_0 = n/r)$ of a hyperfunction on the Shilov boundary. Thus, in the tube case, The Hua operator plays the same role with respect to the Shilov boundary as the algebra $\mathcal{D}(\Omega)^G$ does with respect to the maximal boundary. In his paper [13], Lassalle showed the existence of a smaller system (a projection of the Hua operator) with the same properties.

Later Shimeno [16] generalize the result of Johnson and Korányi; namely he proved that a function is eigenfunction of $\mathcal{H}$ if and only if it is a Poisson transform $\mathcal{P}_s f$ of a hyperfunction on the Shilov boundary for generic $s \in \mathbb{C}$.

In [2], the first author gave a characterization of the Poisson transform $\mathcal{P}_s$ on $L^p(S)$, which closes the tube type symmetric domains case characterization.

It thus arises the question of characterizing the range of the Poisson transform $\mathcal{P}_s$ on $L^p(S)$, $1 < p < +\infty$, for non-tube bounded symmetric domains on $L^p(S)$. The purpose of this paper is to answer this question.

For general bounded symmetric domains the Poisson integrals are not eigenfunctions of the second-order Hua operator $\mathcal{H}$, see for instance [3] or [12]. However for type $\mathbf{I}_{r,r+b}$ domains of non-tube type, (see [3] and [12]) there is a variant of the second-order Hua operator, $\mathcal{H}^{(1)}$, by taking the first component of $\mathcal{H}$, since in this case $\mathfrak{k}_{\mathbb{C}}$ is a sum of two irreducible ideals $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}}^{(1)} \oplus \mathfrak{k}_{\mathbb{C}}^{(2)}$. It is proved, in [12] (and in [3] for the harmonic case, $s = (2r+b)/r$) that a smooth function f on $\mathbf{I}_{r,r+b}$ is a solution of the Hua system, $\mathcal{H}^{(1)} f = \frac{1}{4}(s^2 - (r+b)^2) f I_r$ if and only if it is the Poisson transform $\mathcal{P}_s$ of a hyperfunction on the Shilov boundary.

For general non-tube domains, and for the harmonic case (i.e., for $s = n/r$ in our parametrization) the characterization of the image of the Poisson transform $\mathcal{P}_{\frac{n}{r}}$ on hyperfunctions over the Shilov boundary was done by Berline and Vergne [3] where certain third-order Hua operator was introduced. Recently, the second author and Zhang [12] generalize the result of of Berline and Vergne to any (generic) s. They introduce two third-order Hua operators $\mathcal{U}$ and $\mathcal{W}$ (different from the Berline and Vergne operator) and prove that an eigenfunction f of $\mathcal{D}(\Omega)^G$ is a solution the Hua system (6) if and only if it is a Poisson transform of a hyperfunction on the Shilov boundary.

Let $\mathcal{E}_s(\Omega)$ be the space of harmonic functions on Ω that are solutions of the Hua system (for type $\mathbf{I}_{r,r+b}$ domains, an eigenfunction of $\mathcal{H}^{(1)}$ is indeed harmonic). Then the image $\mathcal{P}_s(L^p(S))$ is a proper closed subspace of $\mathcal{E}_s(\Omega)$. For $1 < p < +\infty$, we introduce the Hua-Hardy type space, $\mathcal{E}^p_s(\Omega)$ of functions $f \in \mathcal{E}_s(\Omega)$ such that

$$\|f\|_{s,p} = \sup_{t>0} e^{-t(\Re(s)r-n)} (\int_K |f(ka_t)|^p dk)^{1/p} < +\infty.$$

Our main result (see Theorem 4.10) says that *if $s \in \mathbb{C}$ is such that $\Re(s) > \frac{a}{2}(r-1)$, a smooth function F on Ω is the Poisson transform $F = \mathcal{P}_s f$ of a function $f \in L^p(S)$ if and only if $f \in \mathcal{E}^p_s(\Omega)$*. Our method of proving this characterization uses an L^2 version of this theorem (see Theorem 4.8) and an inversion formula for the Poisson transform (see Proposition 4.9) which needs Fatou-type theorems (see Theorem 4.3 and Theorem 4.5).

2. Preliminaries

Let Ω be an irreducible bounded symmetric domain in a complex $n-$dimensional space V. Let G be be the identity component of the group of biholomorphic automorphisms of Ω, and K be the isotropy subgroup of G at the point $0 \in \Omega$. Then K is a maximal compact subgroup of G and as a Hermitian symmetric space, $\Omega = G/K$. Let $\mathfrak{g}$ be the Lie algebra of G, and

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be its Cartan decomposition. The Lie algebra $\mathfrak{k}$ of K has one dimensional center $\mathfrak{z}$. Then there exists an element $Z_0 \in \mathfrak{z}$ such that adZ_0 defines the complex structure of $\mathfrak{p}$. Let

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-$$

be the corresponding eigenspaces decomposition of $\mathfrak{g}_{\mathbb{C}}$, the complexification of $\mathfrak{g}$. Let $G_{\mathbb{C}}$ be a connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and P^+, $K_{\mathbb{C}}$, P^- be the analytic subgroups of $G_{\mathbb{C}}$ corresponding to $\mathfrak{p}^+$, $\mathfrak{k}_{\mathbb{C}}$, $\mathfrak{p}^-$. Denote by σ the conjugaison of $G_{\mathbb{C}}$ with respect to G. Then we have, $\sigma(P^{\pm}) \subset P^{\mp}$ and $\sigma(K_{\mathbb{C}}) \subset K_{\mathbb{C}}$.

Let $\mathfrak{h}$ be a maximal Abelian subalgebra of $\mathfrak{k}$, and let $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ be the corresponding set of roots. As Z_0 belongs to $\mathfrak{h}$, the space $\mathfrak{p}^+$ is stable by ad$\mathfrak{h}$. The roots $\gamma \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ such that $\mathfrak{g}^{\gamma} \subset \mathfrak{p}^+$ are said to be positive non-compact, and we denote by Φ the set of such roots. Let $\gamma \in \Phi$, then one may choose elements $H_\gamma \in i\mathfrak{h}$, $E_\gamma \in \mathfrak{g}^{\gamma}$, $E_{-\gamma} \in \mathfrak{g}^{-\gamma}$ such that $[E_\gamma, E_{-\gamma}] = H_\gamma$ and $\sigma(E_\gamma) = -E_{-\gamma}$. Let $X_\gamma = E_\gamma + E_{-\gamma}$ and $Y_\gamma = i(E_\gamma - E_{-\gamma})$. Then, by a classical Harish-Chandra construction, there exists a maximal set $\Gamma = \{\gamma_1, \dots, \gamma_r\}$ of strongly orthogonal roots in Φ. For simplicity, let us set for, $1 \leq j \leq r$,

$$E_j = E_{\gamma_j}, \quad X_j = X_{\gamma_j}, \quad Y_j = Y_{\gamma_j}.$$

Then,

$$\mathfrak{a} = \sum_{j=1}^{r} \mathbb{R} X_j,$$

is a Cartan subspace of the pair $(\mathfrak{g}, \mathfrak{k})$. Let $\mathfrak{a}^*$ denote the dual of $\mathfrak{a}$ and let $\{\beta_1, \beta_2, \dots, \beta_r\}$ be a basis of $\mathfrak{a}^*$ determined by

$$\beta_j(X_k) = 2\delta_{j,k}, \quad 1 \leq j, k \leq n.$$

The restricted root system $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ of $\mathfrak{g}$ relative to $\mathfrak{a}$ is (of type C_r or BC_r) given by

$$\pm\beta_j \quad (1 \leq j \leq r) \quad \text{each with multiplicity } \ 1,$$

$$\pm\frac{1}{2}(\beta_j \pm \beta_k) \quad (1 \leq j \neq k \leq r) \quad \text{each with multiplicity } \ a,$$

and possibly

$$\pm\frac{1}{2}\beta_j \quad (1 \leq j \leq r) \quad \text{each with multiplicity } \ 2b.$$

Let $\Sigma^+ = \{\beta_j, \frac{1}{2}\beta_j, \frac{1}{2}(\beta_\ell \pm \beta_k); \quad 1 \leq j \leq r, \ 1 \leq \ell \neq k \leq r\}$ the set of positive restricted roots. Then the set $\Lambda = \{\alpha_1, \ \dots, \ \alpha_{r-1}, \ \alpha_r\}$ of simple roots in Σ^+ is such that

$$\alpha_j = \frac{1}{2}(\beta_{r-j+1} - \beta_{r-j}), \ 1 \leq j \leq r-1$$

and

$$\alpha_r = \begin{cases} \beta_1 & \text{for tube case} \\ \frac{1}{2}\beta_1 & \text{for non-tube case.} \end{cases}$$

Let $\Lambda_1 = \{\alpha_1, \dots, \alpha_{r-1}\}$ and write $\Sigma_1 = \Sigma \cap \mathbb{Z} \cdot \Lambda_1$. Define

$$\mathfrak{m}_{1,1} = \mathfrak{m} + \mathfrak{a} + \sum_{\gamma \in \Sigma_1} \mathfrak{g}^{\gamma}, \quad \mathfrak{n}_1^+ = \sum_{\gamma \in \Sigma^+ \setminus \Sigma_1} \mathfrak{g}^{\gamma}.$$

Let

$$\mathfrak{a}_1 = \{H \in \mathfrak{a} \ : \ \gamma(H) = 0 \ \forall \gamma \in \Lambda_1\},$$

then $\mathfrak{m}_{1,1}$ is the centralizer of $\mathfrak{a}_1$ in $\mathfrak{g}$ and $\mathfrak{p}_1 = \mathfrak{m}_{1,1} + \mathfrak{n}_1^+$ is a standard parabolic subalgebra of $\mathfrak{g}$ with Langlands decomposition $\mathfrak{m}_1 + \mathfrak{a}_1 + \mathfrak{n}_1^+$, where $\mathfrak{m}_1$ is the orthocomplement of $\mathfrak{a}_1$ in $\mathfrak{m}_{1,1}$ with respect to the Killing form. Note that $\theta(\mathfrak{n}_1^+) = \sum_{\gamma \in \Sigma^+ \setminus \Sigma_1} \mathfrak{g}^{-\gamma}$. Let P_1 be the corresponding parabolic subgroup and $P_1 = M_1 A_1 N_1^+$ its Langlands decomposition. Obviously, P_1 is a maximal parabolic subgroup of G, thus the Shilov boundary S can be viewed as $S = G/P_1 = K/K_1$, where $K_1 = M_1 \cap K$

If we define the element $X_0 = \sum_{j=1}^{r} X_j$, Then $\mathfrak{a}_1 = \mathbb{R}X_0$. Let

$$\mathfrak{a}(1) = \sum_{j=1}^{r-1} \mathbb{R}(X_j - X_{j+1})$$

be the orthocomplement of $\mathfrak{a}_1$ in $\mathfrak{a}$ with respect to the Killing form,

$$\mathfrak{a} = \mathfrak{a}_1 \oplus^{\perp} \mathfrak{a}(1) = \mathbb{R}X_0 \oplus^{\perp} \sum_{j=1}^{r-1} \mathbb{R}(X_j - X_{j+1}). \tag{1}$$

We denote ρ_0 the linear form on $\mathfrak{a}_1$ such that, $\rho_0(X_0) = r$. We extend ρ_0 to $\mathfrak{a}$ via the orthogonal projection (1). If ρ_1 is the restriction of ρ to $\mathfrak{a}_1$, then it is clear that

$$\rho_1(X_0) = rb + r + a\frac{r(r-1)}{2} = n.$$

Again, we extend ρ_1 to $\mathfrak{a}$ via the orthogonal projection (1). Then

$$\rho_1 = (b + 1 + a\frac{(r-1)}{2})\rho_0 = \frac{n}{r}\rho_0.$$

For $g \in G$, define $H(g) \in \mathfrak{a}$ as the unique element such that

$$g \in K \exp(H(g))N \subset KAN = G.$$

We also denote by $\kappa(g) \in K$ and $H_1(g) \in \mathfrak{a}_1$ the unique elements such that

$$g \in \kappa(g) M_1 \exp(H_1(g)) N_1 \subset K M_1 A_1 N_1 = G.$$

The following lemma will be useful for the sequel.

Lemma 2.1 ([15, Lemma 6.1.6]). *(i) Let $x, y \in G$, $\bar{n} \in \bar{N}_1$ and $a \in A_1$. Then*

$$H_1(x\kappa(y)) = H_1(xy) - H_1(y) \tag{2}$$

$$H_1(\bar{n}a^{-1}) = H_1(\bar{n}) - H_1(a) \tag{3}$$

(ii) Let $t > 0$ and $\bar{n} \in \bar{N}_1$. Then

$$\rho_0(H_1(a_t\bar{n}a_{-t})) \leq \rho_0(H_1(\bar{n})). \tag{4}$$

3. The Poisson transform and the Hua operators

For any real analytic manifold X, we denote by $\mathcal{B}(X)$ the space of all hyperfunctions on X. We will view a function on the Shilov boundary $S = G/P_1$ as a P_1-invariant function on G. For $s \in \mathbb{C}$, we denote by $\mathcal{B}(G/P_1; s)$ the space of hyperfunctions f on G satisfying

$$f(gman) = e^{(s\rho_0 - \rho_1)\log a} f(g), \ \ \forall g \in G,\ m \in M_1,\ a \in A_1,\ n \in N_1^+,$$

The Poisson transform of a function $f \in \mathcal{B}(G/P_1; s)$, is defined by

$$\mathcal{P}_s f(gK) = \int_K e^{-\langle s\rho_0 + \rho_1, H_1(g^{-1}k)\rangle} f(k) dk.$$

Since $G = KP_1$, the restriction from G to K defines a $G-$isomorphism from $\mathcal{B}(G/P_1, s)$ onto the space $\mathcal{B}(K/K_1)$ of all hyperfunctions f on K such that $f(kh) = f(k)$ for all $h \in K_1$.

We review the construction of Hua operators of the second order (see [7]) and the third order (see [3], [12]).
Let $\{v_j\}$ be a basis of $\mathfrak{p}^+$ and $\{v_j^*\}$ be th dual basis of $\mathfrak{p}^-$ with respect to the Killing form. Let $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ denote the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. The second-order Hua operator, is the element of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \otimes \mathfrak{k}_{\mathbb{C}}$ defined by

$$\mathcal{H} = \sum_{i,j} v_i v_j^* \otimes [v_j, v_i^*]$$

It is known that the Hua operator does not depends on basis, therefor, for computations one can choose the root vectors basis $\{E_j\}_{j=1}^r$.

For tube domains the Hua operator $\mathcal{H}$ maps the Poisson kernels

$$P_s(gk) = e^{-\langle s\rho_0 + \rho_1, H_1(g^{-1})\rangle}$$

into the center of $\mathfrak{k}_{\mathbb{C}}$, namely the Poisson kernels are its eigenfunctions up to an element in the center, but it is not true for non-tube domains, see [12, Theorem 5.3]. However for non-tube type **I** domains, $\mathbf{I}_{r,r+b} \simeq SU(r, r+b)/S(U(r) \times U(r+b))$, the situation is not quite different from the tube case. In fact, $\mathfrak{k}_{\mathbb{C}}$ is a sum of two irreducible ideals, $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}}^{(1)} + \mathfrak{k}_{\mathbb{C}}^{(2)}$ where

$$\begin{aligned} \mathfrak{k}_{\mathbb{C}}^{(1)} &= \left\{ \begin{pmatrix} A & 0 \\ 0 & \frac{\mathrm{tr}(A)}{r+b} I_{r+b} \end{pmatrix},\ A \in \mathfrak{gl}(r+b, \mathbb{C}) \right\}, \\ \mathfrak{k}_{\mathbb{C}}^{(1)} &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix},\ D \in \mathfrak{sl}(r+b, \mathbb{C}) \right\}. \end{aligned}$$

There is a variant of the Hua operator, $\mathcal{H}^{(1)}$ see [3], [12], by taking the projection of $\mathcal{H}$ onto $\mathfrak{k}_{\mathbb{C}}^{(1)}$. In [12], the second author and Zhang showed that the operator $\mathcal{H}^{(1)}$ has the Poisson kernels as its eigenfunctions and they found the eigenvalues. They proved further that the eigenfunctions of the Hua operator $\mathcal{H}^{(1)}$ are harmonic functions (i.e., eigenfunctions of all invariant differential operators on Ω), and gave the following characterization of the range of the Poisson transform for $\mathbf{I}_{r,r+b}$.

Theorem 3.1 ([12, Theorem 6.1])**.** *Suppose $s \in \mathbb{C}$ satisfies the following condition*

$$-4[b+1+j+\frac{1}{2}(s-r-b)] \notin \{1,2,3,\cdots\}, \quad \textit{for} \ \ j = 0 \textit{ and } 1.$$

Then the Poisson transform $\mathcal{P}_s$ is a $G-$isomorphism of $\mathcal{B}(S)$ onto the space of smooth functions f on Ω that satisfy

$$\mathcal{H}^{(1)} f = \frac{1}{4}(s^2 - (r+b)^2) f I_r. \tag{5}$$

For the characterization of range of the Poisson transform for general non-tube domains the second author and Zhang [12] introduced new third-order Hua operators $\mathcal{U}$ and $\mathcal{W}$:

$$\mathcal{U} = \sum_{i,j,k} v_i^* v_j^* v_k \otimes [v_i, [v_j, v_k^*]],$$

$$\mathcal{W} = \sum_{i,j,k} v_k v_i^* v_j \otimes [[v_k^*, v_i], v_j], .$$

Similarly to $\mathcal{H}$, the operators $\mathcal{U}$ and $\mathcal{W}$ do not depend on the basis.

Denote

$$c = 2(n+1) + \frac{1}{n}(a^2-4)\dim(\mathcal{P}^{(1,1)}),$$

where $\mathcal{P}^{(1,1)}$ is the dimension of the irreducible subspaces of holomorphic polynomials on $\mathfrak{p}^+$ with lowest weight $-\gamma_1 - \gamma_2$. For any $s \in \mathcal{C}$, put $\sigma = \frac{1}{2}(s + \frac{n}{r})$. For general non-tube domains we have the following

Theorem 3.2 ([12, Theorem 7.2])**.** *Let Ω be a bounded symmetric non-tube domain of rank r in $\mathbb{C}^n$. Suppose $s \in \mathbb{C}$ satisfies*

$$-4[b+1+j\frac{a}{2}+\frac{1}{2}(s-\frac{r}{n})] \notin \{1,2,3,\cdots\}, \quad \textit{for} \;\; j=0 \textit{ and } 1.$$

Then the Poisson transform $\mathcal{P}_s$ is a $G-$isomorphism of $\mathcal{B}(S)$ onto the space of harmonic functions f on Ω that satisfy

$$\left(\mathcal{U} - \frac{-2\sigma^2+2p\sigma+c}{\sigma(2\sigma-p-b)}\mathcal{W}\right) f = 0, \tag{6}$$

4. The L^p-range of the Poisson transform

For $1 < p < +\infty$, we will consider the space $L^p(S) = L^p(K/K_1)$ as the space of all complex valued measurable (classes) functions f on K that are K_1-invariant and satisfying

$$\|f\|_p = \left(\int_K |f(k)|^p dk\right)^{1/p} < +\infty,$$

where dk is the Haar measure of K. Let $d\bar{n}$ be the invariant measure on $\bar{N}_1 = \theta(N_1)$ with the normalization

$$\int_{\bar{N}_1} e^{\langle -2\rho_1, H_1(\bar{n})\rangle} d\bar{n} = 1. \tag{7}$$

Then for a continuous function f on S we have

$$\int_K f(k)dk = \int_{\bar{N}_1} f(\kappa(\bar{n}))e^{-2\langle \rho_1, H_1(\bar{n})\rangle} d\bar{n}. \tag{8}$$

The space $L^p(S)$ can be viewed as a subspace of $\mathcal{B}(S)$, thus its image $\mathcal{P}_s(L^p(S))$ is a proper closed subspace of $\mathcal{E}_s(\Omega)$. We will now, for specific s, characterize this image. For this we need some information on the integral $\mathbf{c}_s$ in following proposition.

Proposition 4.1. *For $s \in \mathbb{C}$ such that $\Re(s) > \frac{a}{2}(r-1)$, the integral*

$$\mathbf{c}_s = \int_{\bar{N}_1} e^{-\langle s\rho_0+\rho_1, H_1(\bar{n})\rangle} d\bar{n}$$

converges absolutely to a constant $\mathbf{c}_s \neq 0$.

Proof. For $s \in \mathbb{C}$ let $\lambda_s \in \mathfrak{a}^*_{\mathbb{C}}$ be the linear form defined by

$$\lambda_s(H) = (s\rho_0 - \rho_1)(H_1) + \rho(H), \quad H \in \mathfrak{a}$$

where H_1 is the projection of H onto $\mathfrak{a}_1$. Then the condition $\Re(s) > \frac{a}{2}(r-1)$ is equivalent to

$$\Re(\langle \lambda_s, \alpha\rangle) > 0 \;\; \forall \alpha \in \Sigma^+ \setminus \Sigma_1. \tag{9}$$

Moreover, we can choose (see for example [15, Lemma 6.1.4]) ω in the Weyl group W of Σ such that

$$\begin{array}{rl} (i) & \omega\cdot H = H, \quad \forall H\in\mathfrak{a}_1,\\ (ii) & \omega(\Sigma^+\cap\Sigma_1) = -\Sigma^+\cap\Sigma_1,\\ (iii) & \omega(\Sigma^+\setminus\Sigma_1) = \Sigma^+\setminus\Sigma_1. \end{array}$$

Since $\langle\omega\lambda_s,\alpha\rangle = \langle\lambda_s,\omega^{-1}\alpha\rangle$, the condition (9) is equivalent to

$$\Re(\langle\omega\lambda_s,\alpha\rangle) > 0, \quad \forall\alpha\in\Sigma^+$$

Furthermore

$$\langle s\rho_0+\rho_1, H_1(g)\rangle = \langle\omega\lambda_s+\rho, H(g)\rangle$$

so that

$$\int_{\bar N_1} e^{-\langle s\rho_0+\rho_1,H_1(\bar n)\rangle}d\bar n = \int_{\bar N_1} e^{-\langle\omega\lambda_s+\rho,H(\bar n)\rangle}d\bar n$$

and the right hand side is the Harish-Chandra c function, $c(\omega\lambda_s)$ associated with the maximal parabolic subgroup, which converges absolutely, see [15].

□

Let $s\in\mathbb{C}$. Let $\mathcal{E}_s(\Omega)$ be the space of harmonic functions on Ω that satisfy (5) in type I domains or (6) in general domains. It is clear that the image $\mathcal{P}_s(L^p(S))$ is a proper closed subspace of the eigenspace $\mathcal{E}_s(\Omega)$. Hence, it is natural to look for a characterization of those $F\in\mathcal{E}_s(\Omega)$ that are Poisson transform of some $f\in L^p(S)$.

For any $1<p<\infty$, let $\mathcal{E}^p_s(\Omega)$ denote the Hua-Hardy type space of functions $f\in\mathcal{E}_s(\Omega)$ such that

$$\|f\|_{s,p} = \sup_{a\in A_1} e^{-\langle\Re(s)\rho_0-\rho_1,\log a\rangle}\Big(\int_K |f(ka)|^p dk\Big)^{1/p} < +\infty.$$

Since $\mathfrak{a}_1 = \mathbb{R}X_0$, the above integral becomes

$$\|f\|_{s,p} = \sup_{t>0} e^{-t(\Re(s)r-n)}\Big(\int_K |f(ka_t)|^p dk\Big)^{1/p},$$

where $a_t = \exp(tX_0)$.

4.1. **Fatou type theorems.**

As a preparation to Fatou-type theorems we prove the following

Proposition 4.2. *Let $s\in\mathbb{C}$ be such that $\Re(s) > \frac{a}{2}(r-1)$. Let Ψ_t be the function defined on $\bar N_1$ by*

$$\Psi_t(\bar n) = e^{-\langle s\rho_0+\rho_1,H_1(\bar n)\rangle + \langle s\rho_0-\rho_1,H_1(a_t\bar n a_{-t})\rangle}.$$

Then there exists a non-negative function $\Phi\in L^1(\bar N_1)$ such that $\Psi_t\le\Phi$ for each t.

Proof. It follows from (4), that for any $t > 0$ and for any $\bar{n} \in \bar{N}_1$,

$$0 \leq \rho_0(H_1(a_t \bar{n} a_{-t}) \leq \rho_0(H_1(\bar{n}).$$

Therefore,

$$|\Psi_t(\bar{n})| \leq \begin{cases} e^{-\langle \Re(s)\rho_0 + \rho_1, H_1(\bar{n})\rangle} & \text{if } \frac{a}{2}(r-1) < \Re(s) \leq \frac{a}{2}(r-1) + b + 1 \\ e^{-2\langle \rho_1, H_1(\bar{n})\rangle} & \text{if } \Re(s) > \frac{a}{2}(r-1) + b + 1 \end{cases}$$

and the second hand is an integrable function on $\bar{N}_1$ by (7) and Proposition 4.1 .

□

Let $\mathcal{C}(S)$ be the space of complex-valued continuous functions on S with the topology of uniform convergence.

Theorem 4.3. *Let $s \in \mathbb{C}$ be such that $\Re(s) > \frac{a}{2}(r-1)$. Then*

$$f(k) = \mathbf{c}_s^{-1} \lim_{t \to +\infty} e^{-(rs-n)t} \mathcal{P}_s f(ka_t)$$

uniformly, for $f \in \mathcal{C}(S)$.

Proof. Let $f \in \mathcal{C}(S)$, then

$$\mathcal{P}_s f(ka_t) = \int_K e^{-\langle s\rho_0 + \rho_1, H_1(a_{-t}h)\rangle} f(kh) dh.$$

We transform this integral using the formula (8) to an integral over $\bar{N}_1$,

$$\mathcal{P}_s f(ka_t) = \int_{\bar{N}_1} e^{-\langle s\rho_0 + \rho_1, H_1(a_{-t}\kappa(\bar{n}))\rangle} f(k\kappa(\bar{n})) e^{-2\langle \rho_1, H_1(\bar{n})\rangle} d\bar{n},$$

and by (2) we get

$$\mathcal{P}_s f(ka_t) = \int_{\bar{N}_1} e^{-\langle s\rho_0 + \rho_1, H_1(a_{-t}\bar{n})\rangle} e^{\langle s\rho_0 - \rho_1, H_1(\bar{n})\rangle} f(k\kappa(\bar{n})) d\bar{n}.$$

which by the substitution $\bar{n} \mapsto a_{-t}\bar{n}a_t$ and (3), becomes

$$\mathcal{P}_s f(ka_t) = e^{\langle s\rho_0 - \rho_1, H_1(a_t)\rangle} \times$$
$$\int_{\bar{N}_1} e^{-\langle s\rho_0 + \rho_1, H_1(\bar{n})\rangle + \langle s\rho_0 - \rho_1, H_1(a_t \bar{n} a_{-t})\rangle} f(k\kappa(a_t \bar{n} a_{-t})) d\bar{n}.$$

But $\rho_1 = \frac{n}{r}\rho_0$, and $a_t \bar{n} a_{-t} \to e$ when $t \to +\infty$, thus, by Proposition 4.2,

$$\lim_{t \to +\infty} e^{-(rs-n)t} \mathcal{P}_s f(ka_t) = \mathbf{c}_s f(k).$$

□

Let

$$\varphi_s(a_t) := \int_K e^{-\langle s\rho_0+\rho_1, H(a_{-t}k)\rangle} dk.$$

then, it follows from the above theorem that

$$\lim_{t\to\infty} e^{-(rs-n)t}\varphi_s(a_t) = \mathbf{c}_s, \quad \text{if } \Re(s) > \frac{a}{2}(r-1). \tag{10}$$

As a consequence we can prove the following

Proposition 4.4. *Let $s \in \mathbb{C}$ be such that $\Re(s) > \frac{a}{2}(r-1)$. Then there exists a positive constant γ_s such that, for $1 < p < \infty$ and $f \in L^p(S)$, we have*

$$\left(\int_K |\mathcal{P}_s f(ka_t)|^p dk\right)^{1/p} \leq \gamma_s\, e^{(rs-n)t} \|f\|_p.$$

Proof. For $t > 0$, we define the function p_s^t by

$$p_s^t(k) = e^{-\langle s\rho_0+\rho_1, H_1(a_{-t}k^{-1})\rangle}, \quad k \in K.$$

Then the Poisson transform can be written as the convolution

$$\mathcal{P}_s f(ka_t) = (f * p_s^t)(k).$$

Hence, to prove the proposition we use the Haussedorf-Young inequality,

$$\left(\int_K |\mathcal{P}_s f(ka_t)|^p dk\right)^{1/p} \leq \|p_s^t\|_1 \, \|f\|_p \qquad (p > 1),$$

and (10).

□

Let, as usual, $\hat{K}$, be the set of equivalence classes of finite dimensional irreducible representations of K. For $\delta \in \hat{K}$, let $\mathcal{C}(S)_\delta$ be the linear span of all $K-$finite vectors on S of type δ. It is well known that the space $\mathcal{C}^K(S) := \oplus_{\delta\in\hat{K}}\mathcal{C}(S)_\delta$ is dense in $\mathcal{C}(S)$. Recall also, that the space $\mathcal{C}(S)$ is dense in $L^p(S)$ for $1 < p < \infty$.

Theorem 4.5. *Let $s \in \mathbb{C}$ be such that $\Re(s) > \frac{a}{2}(r-1)$. Then*

$$f(k) = \mathbf{c}_s^{-1} \lim_{t\to+\infty} e^{-(rs-n)t}\mathcal{P}_s f(ka_t)$$

in $L^p(S)$, for $1 < p < \infty$.

Proof. Let $f \in L^p(S)$. By the above density arguments, for any $\epsilon > 0$, there exists $\varphi \in \mathcal{C}^K(S)$ such that $\|f - \varphi\|_p < \epsilon$. Then we have

$$\begin{aligned}\|\mathbf{c}_s^{-1}e^{-(rs-n)t}P_s^t f - f\|_p &\leq \|\mathbf{c}_s^{-1}e^{-(rs-n)t}P_s^t(f-\varphi)\|_p + \\ &+ \|\mathbf{c}_s^{-1}e^{-(rs-n)t}P_s^t\varphi - \varphi\|_p + \|\varphi - f\|_p\end{aligned}$$

where the function $P_s^t f$ is defined by

$$P_s^t f(k) = \mathcal{P}_s f(ka_t). \tag{11}$$

By Proposition 4.4,

$$\|\mathbf{c}_s^{-1} e^{-(rs-n)t} P_s^t(f-\varphi)\|_p \leq \gamma_s |\mathbf{c}_s^{-1}| \|f-\varphi\|_p,$$

and by Theorem 4.3

$$\lim_{t\to+\infty} \|\mathbf{c}_s^{-1} e^{-(rs-n)t} P_s^t \varphi - \varphi\|_p = 0.$$

Thus, $\lim_{t\to+\infty} \|\mathbf{c}_s^{-1} e^{-(rs-n)t} P_s^t f - f\|_p \leq \epsilon(\gamma_s + 1)$ and this proves the theorem. $\square$

We can now prove the following estimates

Proposition 4.6. *Let $s \in \mathbb{C}$ be such that $\Re(s) > \frac{a}{2}(r-1)$. Then there exists a positive constant γ_s such that for $1 < p < +\infty$ and $f \in L^p(S)$,*

$$|\mathbf{c}_s| \|f\|_p \leq \|\mathcal{P}_s f\|_{s,p} \leq \gamma_s \|f\|_p. \tag{12}$$

Proof. In fact, the right hand side of the estimate (12) follows from Proposition 4.4. On the other hand, by Theorem 4.5 we have

$$\lim_{t\to\infty} e^{(n-rs)t} \mathcal{P}_s f(ka_t) = \mathbf{c}_s f(k)$$

in $L^p(S)$. Hence, there exists a sequence $(t_j)_j$, with $t_j \to +\infty$ when $j \to +\infty$ such that $\lim_{j\to+\infty} e^{(n-rs)t_j} \mathcal{P}_s f(ka_{t_j}) = \mathbf{c}_s f(k)$, almost every where in K. Therefore, by the classical Fatou lemma,

$$|\mathbf{c}_s| \|f\|_p \leq \sup_j e^{(n-r\Re(s))t_j} \|P_s^{t_j} f\|_p$$

and this is how we prove the left hand side of (12). $\square$

4.2. **The L^2−Poisson transform range.** Recall that

$$L^2(S) = \oplus_{\delta\in\hat{K}} V_\delta$$

where V_δ is the finite linear span of $\{\varphi_\delta \circ k, \ \ k \in K\}$, where φ_δ is the zonal spherical function corresponding to δ.
For $s \in \mathbb{C}$ and $\delta \in \hat{K}$, define the *generalized spherical function* $\Phi_{s,\delta}$ on A_1 by

$$\Phi_{s,\delta}(a_t) = (\mathcal{P}_s \varphi_\delta)(a_t).$$

Proposition 4.7. *Let $s \in \mathbb{C}$, $\delta \in \hat{K}$ and $f \in V_\delta$. Then for any $k \in K$ and $a_t \in A_1$,*

$$(\mathcal{P}_s f)(ka_t) = \Phi_{s,\delta}(a_t) f(k).$$

Proof. Since M_1 centralize A_1, we can view the operator (11) as a bounded operator on $L^2(S)$. Moreover, P_s^t commutes with the left regular representation of K in $L^2(S)$. Hence, by Schur's lemma, $P_s^t = \Phi_{s,\delta}(a_t) \cdot I$ on each V_δ and the proposition follows. $\square$

The first main theorem of this section can now be stated as follows :

Theorem 4.8. *Let $s \in \mathbb{C}$ be such that $\Re(s) > \frac{a}{2}(r-1)$. A smooth function F on Ω is the Poisson transform $F = \mathcal{P}_s f$ of a function $f \in L^2(S)$ if and only if $F \in \mathcal{E}_s^2(\Omega)$.*

Proof. The necessary condition follows from Proportion 4.6 and [12, Theorem 6.1 and Theorem 7.2]. On the other hand, let $F \in \mathcal{E}_s^p(\Omega)$. We apply again [12, Theorem 6.1 and Theorem 7.2]. Then, there exists a hyperfunction $f \in \mathcal{B}(S)$ such that $F = \mathcal{P}_s f$. Let $f = \sum_{\delta \in \hat{K}} f_\delta$ be its $K-$type decomposition. By Proposition 4.7 we can write

$$F(ka_t) = \sum_{\delta \in \hat{K}} \Phi_{s,\delta}(a_t) f_\delta(k)$$

in $\mathcal{C}^\infty(K \times [0, +\infty[)$.
Now observe that

$$\|F\|_{s,2}^2 = \sup_{t>0} e^{2(n-r\Re(s))t} \sum_{\delta \in \hat{K}} |\Phi_{s,\delta}(a_t)|^2 \|f_\delta\|_2^2 < \infty.$$

Then, if Λ is an arbitrary finite subset of $\hat{K}$, we get

$$e^{2(n-r\Re(s))t} \sum_{\delta \in \Lambda} |\Phi_{s,\delta}(a_t)|^2 \|f_\delta\|_2^2 \leq \|F\|_{s,2}^2$$

for every $t > 0$ and hence form Theorem 4.3 it follows immediately that

$$|\mathbf{c}_s|^2 \sum_{\delta \in \Lambda} \|f_\delta\|_2^2 \leq \|F\|_{s,2}^2$$

which implies that $f = \sum_{\delta \in K} f_\delta$ in $L^2(S)$ and that

$$|\mathbf{c}_s|^2 \|f\|_2 \leq \|F\|_{s,2}.$$

This ends the proof of the theorem. $\square$

In the following proposition we show how to recover a function $f \in L^2(S)$ from its Poisson transform $\mathcal{P}_s f$.

Proposition 4.9. *Let $s \in \mathbb{C}$ be such that $\Re(s) > \frac{a}{2}(r-1)$. Let $F \in \mathcal{E}_s^2(\Omega)$ and $f \in L^2(S)$ its boundary value. Then the following inversion formula*

$$(13) \quad f(k) = |\mathbf{c}_s|^{-2} \lim_{t\to\infty} e^{2(n-r\Re(s))t} \int_K \overline{e^{-\langle s\rho_0+\rho_1, H_1(a_{-t}k^{-1}h)\rangle}} F(ha_t) dh$$

holds in $L^2(S)$.

Proof. Let $F \in \mathcal{E}_s^2(\Omega)$, then its follows from Theorem 4.8 that there exists a unique $f \in L^2(S)$ such that $F = \mathcal{P}_s f$. Let $f = \sum_{\delta\in\hat{K}} f_\delta$ be its $K-$type expansion, then similarly to the preceding proof, we get

$$F(ka_t) = \sum_{\delta\in\hat{K}} \Phi_{s,\delta}(a_t) f_\delta(k). \tag{14}$$

For any $t > 0$, define the complex-valued function on K by

$$g_t(ka_t) = |\mathbf{c}_s|^{-2} e^{2(n-rs)t} \int_K \overline{e^{-\langle s\rho_0+\rho_1, H_1(a_{-t}k^{-1}h)\rangle}} F(ha_t) dh.$$

Next, using the above series expansion we can write according to Theorem 4.5,

$$g_t(h) = |\mathbf{c}_s|^{-2} e^{2(n-rs)t} \sum_{\delta\in\hat{K}} |\Phi_{s,\delta}(a_t)|^2 f_\delta(h).$$

Thus,

$$\|g_t - f\|_2^2 = \sum_{\delta\in\hat{K}} \left| |\mathbf{c}_s|^{-2} e^{2(n-rs)t} |\Phi_{s,\delta}(a_t)|^2 - 1 \right|^2 \|f_\delta\|_2^2,$$

which shows that $\|g_t - f\|_2 \to 0$, since $\lim_{t\to\infty} e^{(n-rs)t}\Phi_{s,\delta}(a_t) = \mathbf{c}_s$. $\square$

4.3. **The** L^p-**Poisson transform range,** $p \neq 2$. We shall now prove the second main result of this paper, more precisely, we shall characterize the L^p-range of the Poisson transform.We will need the following notation. For each function f on Ω, define the function f^t, $t > 0$, on K by

$$f^t(k) = f(ka_t).$$

Theorem 4.10. *Let* $s \in \mathbb{C}$ *be such that* $\Re(s) > \frac{a}{2}(r-1)$. *A smooth function* F *on* Ω *is the Poisson transform* $F = \mathcal{P}_s f$ *of a function* $f \in L^p(S)$ *if and only if* $F \in \mathcal{E}_s^p(\Omega)$.

Proof. We will follow the technique used by Korànyi [9]. Let $(\chi_n)_n$ be an approximation of the identity in $\mathcal{C}(K)$. That is $\chi_n \geq 0$, $\int_K \chi_n(k)dk = 1$ and $lim_{n\to+\infty} \int_{K\setminus U} \chi_n(k)dk = 0$ for every neighborhood U of e in K. Let $F \in \mathcal{E}_s^p(\Omega)$. For each n, define the function F_n on Ω by

$$F_n(gK) = \int_K \chi_n(k) F(k^{-1}g) dk.$$

Then $(F_n)_n$ converges point-wise to F, and since the set $\mathcal{E}_s(\Omega)$ of harmonic functions satisfying the Hua system is $G-$invariant, $F_n \in \mathcal{E}_s(\Omega)$, for each n. Furthermore,

$$F_n^t(ka_tK) = (\chi_n * F^t)(k)$$

and this shows

$$\|F_n^t\|_2 \leq \|\chi_n\|_2 \|F^t\|_p, \tag{15}$$

and

$$\|F_n^t\|_p \leq \|F^t\|_p. \tag{16}$$

It follows from (15)

$$\sup_{t>0} e^{(n-rs)t} \left(\int_K |F_n(ka_t)|^2 dk \right)^{1/2} \leq \|\chi_n\|_2 \|F\|_{s,p}.$$

Thus $F_n \in \mathcal{E}_{s,2}(\Omega)$ and by Theorem 4.8, there exists $f_n \in L^2(S)$ such that $F_n = \mathcal{P}_s f_n$. Now, our goal is to prove that f_n belongs to $L^p(S)$. Using the inversion formula (13) we can write in $L^2(S)$,

$$f_n(k) = \lim_{t \to +\infty} g_n^t(k)$$

where

$$g_n^t(h) = g_n(ha_t) = |\mathbf{c}_s|^{-2} e^{2(n-r\Re(s))t} \int_K \overline{e^{-\langle s\rho_0+\rho_1, H_1(a_{-t}k^{-1}h)\rangle}} F_n(ka_t) dk.$$

Let $\varphi \in \mathcal{C}(S)$ be a continuous function on S, then

$$\int_K f_n(h)\varphi(h) dh = \lim_{t \to \infty} \int_K g_n^t(h)\varphi(h) dh.$$

Moreover,

$$\begin{aligned}
\int_K g_n^t(h)\varphi(h) dh &= |\mathbf{c}_s|^{-2} e^{2(n-r\Re(s))t} \times \\
&\times \int_K \int_K F_n(ka_t)\varphi(h) \overline{e^{-\langle s\rho_0+\rho_1, H_1(a_{-t}k^{-1}h)\rangle}} dk dh \\
&= |\mathbf{c}_s|^{-2} e^{2(n-r\Re(s))t} \int_K \overline{\mathcal{P}_s \bar{\varphi}(ka_t)} F_n(ka_t) dk.
\end{aligned}$$

By the Holder inequality, if q is such that $1/p + 1/q = 1$, we get

$$\begin{aligned}
\left| \int_K g_n^t(h)\varphi(h) dh \right| &\leq |\mathbf{c}_s|^{-2} e^{2(n-r\Re(s))t} \|\mathcal{P}_s \varphi\|_q \|F_n^t\|_p, \\
&\leq |\mathbf{c}_s|^{-2} e^{2(n-r\Re(s))t} \|\mathcal{P}_s \varphi\|_q \|F^t\|_p,
\end{aligned}$$

where the second inequality follows from (16). But $F \in \mathcal{E}_{s,p}(\omega)$, then

$$\left| \int_K g_n^t(h)\varphi(h) dh \right| \leq |\mathbf{c}_s|^{-2} e^{2(n-r\Re(s))t} \|\mathcal{P}_s \varphi\|_q \|F\|_{s,p}.$$

Therefore, by Theorem 4.3,

$$\left|\int_K f_n(h)\varphi(h)dh\right| \leq |\mathbf{c}_s|^{-1}\|\varphi\|_q\|F\|_{s,p},$$

and by taking the supremum over $\varphi \in \mathcal{C}(S)$ with $\|\varphi\|_q = 1$, we get

$$\|f_n\|_p \leq |\mathbf{c}_s|^{-1}\|F\|_{s,p}.$$

Now, for each $\varphi \in L^q(S)$, define the functional

$$T_n(\varphi) = \int f_n(h)\varphi(h)dk.$$

Then it is obvious by (4.3) that

$$|T_n)\varphi)| \leq |\mathbf{c}_s|^{-1}\|\varphi\|_q\|F\|_{s,p}$$

hence, T_n is uniformly bounded operator in $L^q(S)$ with $\sup_n \|T_n\| \leq |\mathbf{c}_s|^{-1}\|F\|_{s,p}$. Thanks to Banach-Alaouglu-Bourbaki's theorem, there exists a subsequence of bounded operators $(T_{n_j})_j$ which converges as $n_j \to +\infty$ to a bounded operator T in $L^q(S)$, under the $*-$weak topology, with $\|T\| \leq |\mathbf{c}_s|^{-1}\|F\|_{s,p}$. Then, by the Riesz representation theorem, there exists a unique function $f \in L^p(S)$ such that

$$T(\varphi) = \int_K f(h)\varphi(h)dh, \ \ \forall \varphi \in L^q(S)$$

with

$$\|f\|_p \leq \|T_n\| \leq |\mathbf{c}_s|^{-1}\|F\|_{s,p}. \tag{17}$$

Now, observe that

$$F_{n_j}(g) = T_{n_j}(e^{-\langle s\rho_0+\rho_1, H_1(g^{-1}k\rangle}),$$

thus, by taking the limit as $n \to +\infty$ we get

$$F(g) = T(e^{-\langle s\rho_0+\rho_1, H_1(g^{-1}k\rangle}) = \mathcal{P}_s f(g)$$

with $|\mathbf{c}_s|\|f\|_p \leq \|F\|_{s,p}$, by (17), and this finishes the proof of the theorem. □

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